5. Seiberg-Witten curve

Recall
$$\begin{cases} Z^{in}(\varepsilon_1,\varepsilon_2,a,m,\Lambda) = \sum_{n=0}^{\infty} \Lambda^{3n} \int_{M(n)} e(\mathcal{V}\otimes e^m) \\ \log Z^{in} = \frac{1}{\varepsilon_1\varepsilon_2}\left(F_0^{in} + \varepsilon_1\varepsilon_2 A^{in} + \frac{\varepsilon_1^2+\varepsilon_2^2}{3} B^{in} + \text{higher}\right) \end{cases}$$

For a while, we do not put the condition $a=m$.

We make a change of variables a (or a^2) $\rightsquigarrow u$
$$u := a^2 - \frac{1}{3}\frac{\partial F_0^{in}}{\partial \log\Lambda}$$
"quantum correction to a^2"

$$\left(\text{cf. } \int_{M(n)} \mu([0]) \cap e(\mathcal{V}\otimes e^m)\right) \qquad c_2(\mathcal{E})/[0] = a^2 + n\varepsilon_1\varepsilon_2$$

$c_2(\mathcal{E})$

1st term

Th.

1) (Nekrasov conj.) [NY, NO, (BE)]

F_0^{in} and derivatives are given by periods of a family of elliptic curves

$$E_{u,m} : y^2 = 4x^3 - \left(\frac{4}{3}u^2 - 4m\Lambda^3\right)x + \left(\frac{8}{27}u^3 - \frac{4}{3}um\Lambda^3 + \Lambda^6\right) \qquad \text{(Seiberg-Witten curve)}$$

e.g. $a = a(u, m, \Lambda)$ is given by $-2\pi\sqrt{-1}\,\dfrac{da}{du} = \omega = \displaystyle\int_A \frac{dx}{y}$

(etc)

including perturb. part

$$\frac{\partial^2 F_0}{\partial a^2} = -2\pi\sqrt{-1}\,\tau = -2\pi\sqrt{-1} \int_B \frac{dx}{y} \Big/ \int_A \frac{dx}{y}$$

2) [NY] $\exp A = \left(-\dfrac{\sqrt{-1}}{\Lambda}\dfrac{du}{da}\right)^{1/2}$

$\exp B = \sqrt{-1}\,\Lambda^{-3/2}\,\Delta^{1/8}$

Δ: discriminant of $E_{u,m}$

<u>Rem</u>. This is very similar to mirror symmetry : GW inv. = periods of mirror

$a \rightsquigarrow u$: mirror transform

(Idea of the proof of [NY])

Consider $p : \hat{\mathbb{C}}^2 \to \mathbb{C}^2$ blow-up at 0

$\cup C$: exceptional curve

$\hat{M}(n)$: moduli space of framed sheaves (E, φ) on $\hat{\mathbb{P}}^2 = \hat{\mathbb{C}}^2 \cup \ell_\infty$

$\mu(C) \subset \hat{M}(n)$ divisor given by $E|_C \not\cong \mathcal{O}_C^{\oplus 2}$

$$\hat{Z}^{in}(\varepsilon_1, \varepsilon_2, a, m ; t, \Lambda) := \sum \Lambda^{3n} \int_{\hat{M}(n)} \exp(t\mu(C)) \cap e(\mathcal{V} \otimes e^m)$$

$\hat{M}(n) \curvearrowleft T^3$ fixed pts : $E \cong I_1(kC) \oplus I_2(-kC)$

$k \in \mathbb{Z}$

I_1, I_2 : ideal sheaves $\curvearrowleft T^2$

$$\Rightarrow \quad \hat{Z}^{in}(\varepsilon_1, \varepsilon_2, a, m ; t, \Lambda) = \sum_{k \in \mathbb{Z}} (\text{contribution from } \mathcal{O}(kC)) \times \underset{\substack{\uparrow \\ 0}}{Z^{in}} \times \underset{\substack{\uparrow \\ \infty}}{Z^{in}} \quad \text{in } C = \mathbb{P}^1$$

$$\Rightarrow \quad \lim_{\varepsilon_1, \varepsilon_2 \to 0} \frac{\hat{Z}^{in}(\varepsilon_1, \varepsilon_2, a, m ; t, \Lambda)}{Z^{in}(\varepsilon_1, \varepsilon_2, a, m : \Lambda)} = \exp(\text{explicit})(\theta\text{-function}) \text{ where } \tau = -\frac{1}{2\pi\sqrt{-1}} \frac{\partial^2 F_0}{\partial a^2}$$

We write $E_\tau = \mathbb{C}/\mathbb{Z}\omega \oplus \mathbb{Z}\omega\tau$ with $\omega = -2\pi\sqrt{-1} \dfrac{da}{du}$

Now E_τ : $y^2 = 4x^3 - g_2 x - g_3$ by $\wp$-function $y = \wp'$, $x = \wp$

Rewrite θ by σ. So $\displaystyle\lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{\hat{Z}^{in}_{q=c}}{Z} = \exp(\text{quad. in } t)\,\sigma(t)$

To compute g_2, g_3

Recall $\sigma(t) = t - \frac{g_2}{2}\frac{t^5}{5!} - 6g_3\frac{t^7}{7!} + \cdots$.

determine lower coefficients

So we study expansions in t in the blowup formula.

Structure thm

$\dfrac{\hat{Z}^{in}(\varepsilon_1,\varepsilon_2,a,m;t,\Lambda)}{Z^{in}(\varepsilon_1,\varepsilon_2,a,m;\Lambda)}$ is a formal power series in t with coeff. in $\mathbb{C}[m,\varepsilon_1,\varepsilon_2,u,\Lambda]$

proof is by relation of $M(n)$ & $\hat{M}(n)$ via wall-crossing.

By the degree reason enough to compute $\int_{\hat{M}(n)}$ with small n

$\longrightarrow$ calculate g_2, g_3

$\deg e(\mathcal{V}\otimes e^m)\cap[M(n)] = \underbrace{3n}_{0}$ is important

One fund. matter theory is asymptotically free.

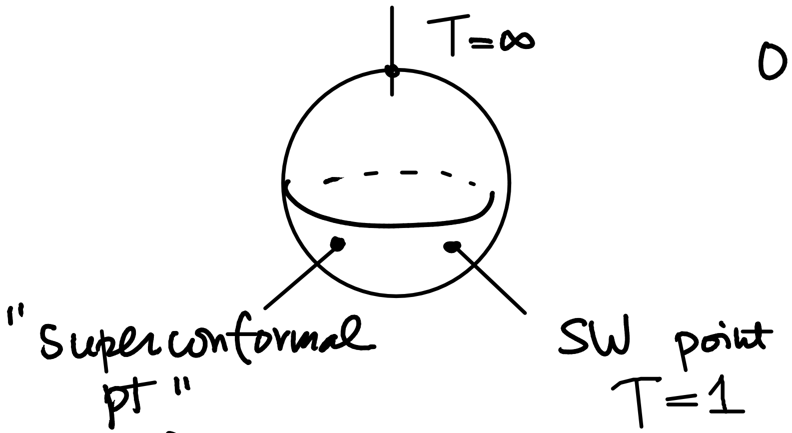
6. Final computation

★ Specialization at $a = m \Rightarrow E_{u,m}$ degenerate

σ-function $\longrightarrow$ sin & exp   easy!

★ In our calculation, we use $T := -\frac{1}{9}\frac{\partial^2 F_0}{\partial(\log\Lambda)^2} = \frac{1}{3}\left(u - \frac{1}{4}\left(\frac{\partial u}{\partial a}\right)^2\right)$
as a variable   (contact term)

It appears with (α^2) in our formula
We use the residue thm



$$0 = \underset{T=\infty}{\mathrm{Res}} + \underset{\substack{T=SW\\=1}}{\mathrm{Res}} + \underset{T=SC}{\mathrm{Res}}$$

Both A&B cycles degenerate

pole of differential

- $T = \infty$ is a pole of high order
  $\longrightarrow$ impossible to compute Res directly
- $T = 1$ simple pole
  $\longrightarrow$ easy to compute Res

$T = sc$ gives a contribution, but

Prop $\sum_{\mathfrak{s}_1} SW(\tilde{\mathfrak{s}}_1) \operatorname{Res}_{u=sc} \mathcal{F} \frac{da}{du} du = 0$

more concretely
$SW(\alpha) := \sum SW(\tilde{\mathfrak{s}}_1)(-1)^{(K_X, K_X+\tilde{\mathfrak{s}}_1)/2} \exp(\tilde{\mathfrak{s}}_1, \alpha)$
has zero of order $\geq \chi_h(X) - (K_X^2) - 3$
at $\alpha = 0$
[Mariño - Moore - Peradze]

This is a nontrivial constraint on SW invariants.

2 proofs

— can be checked for cpx proj. surfaces

— $\mathcal{D}^3 = \mathcal{D}^{3+2c_1(L)}$ up to sign $\Rightarrow$ Prop

Rem. All examples of 4-mfds satisfy this condition

(sc simple type)